Analog-to Digital Conversion Revisited in the Frequency Domain

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Can it be possible to make

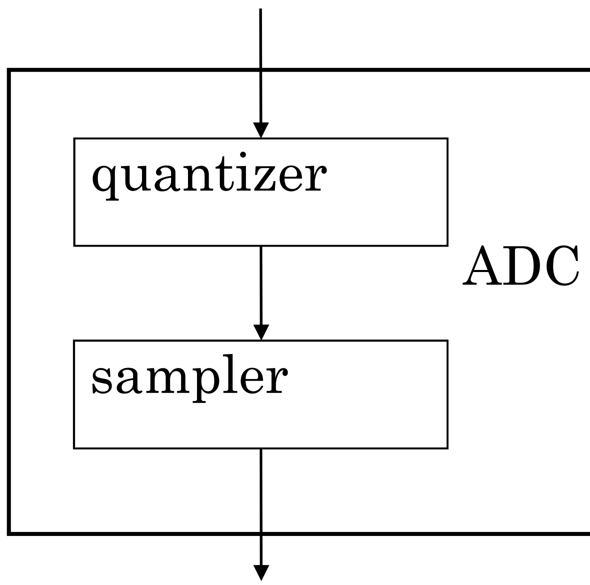
Analog-to-Digital Conversion

noiseless?

Yes or no ?

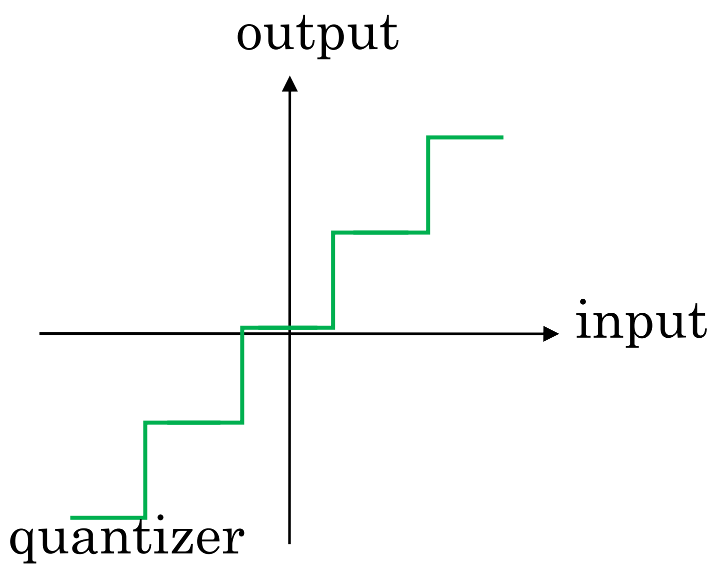
Analog-to-Digital conversion (ADC) is comprised of two steps.

- One is sampling
- And the other is quantization.



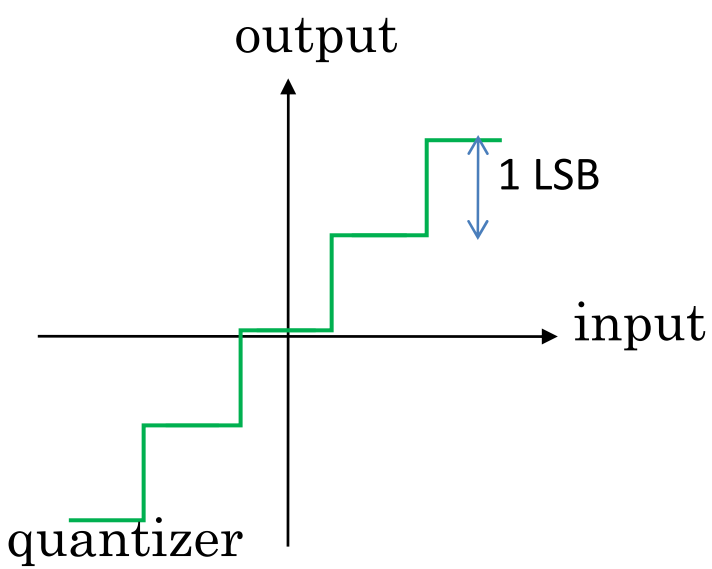
A quantizer is a non-linear circuit

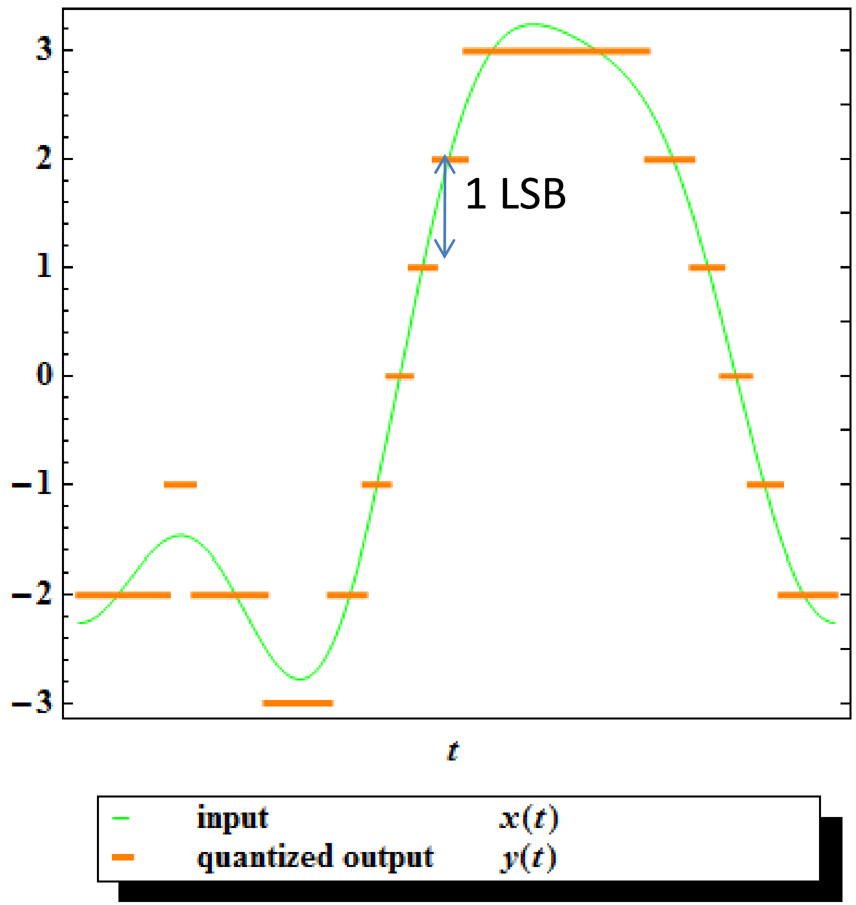
- It has stair case input-output characteristics.



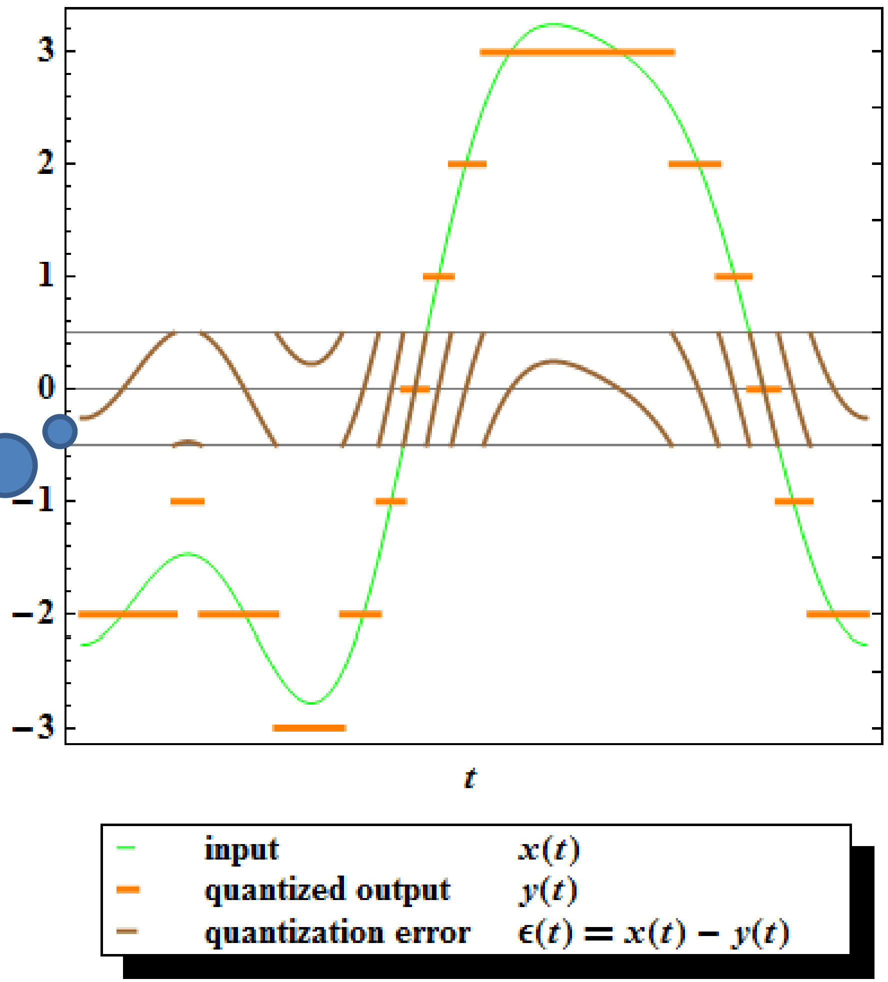
Let us view quantization in the time-domain

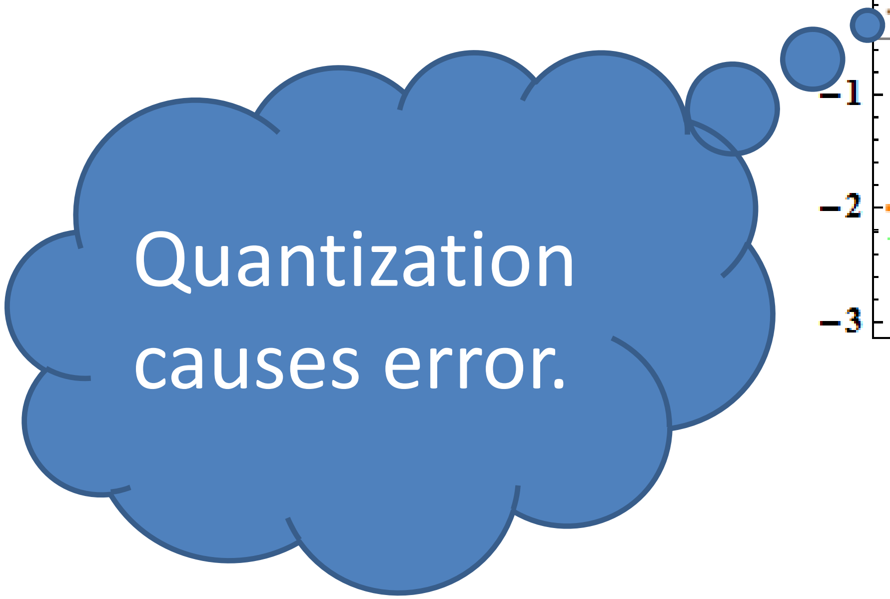
- Output is split into discrete levels separated by 1 LSB.
  - LSB = least significant bit





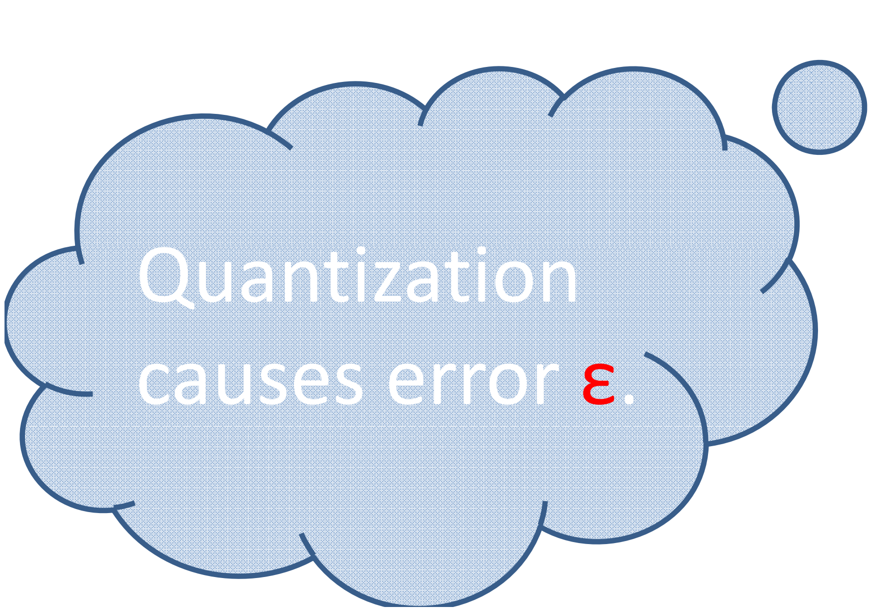
Quantization causes error

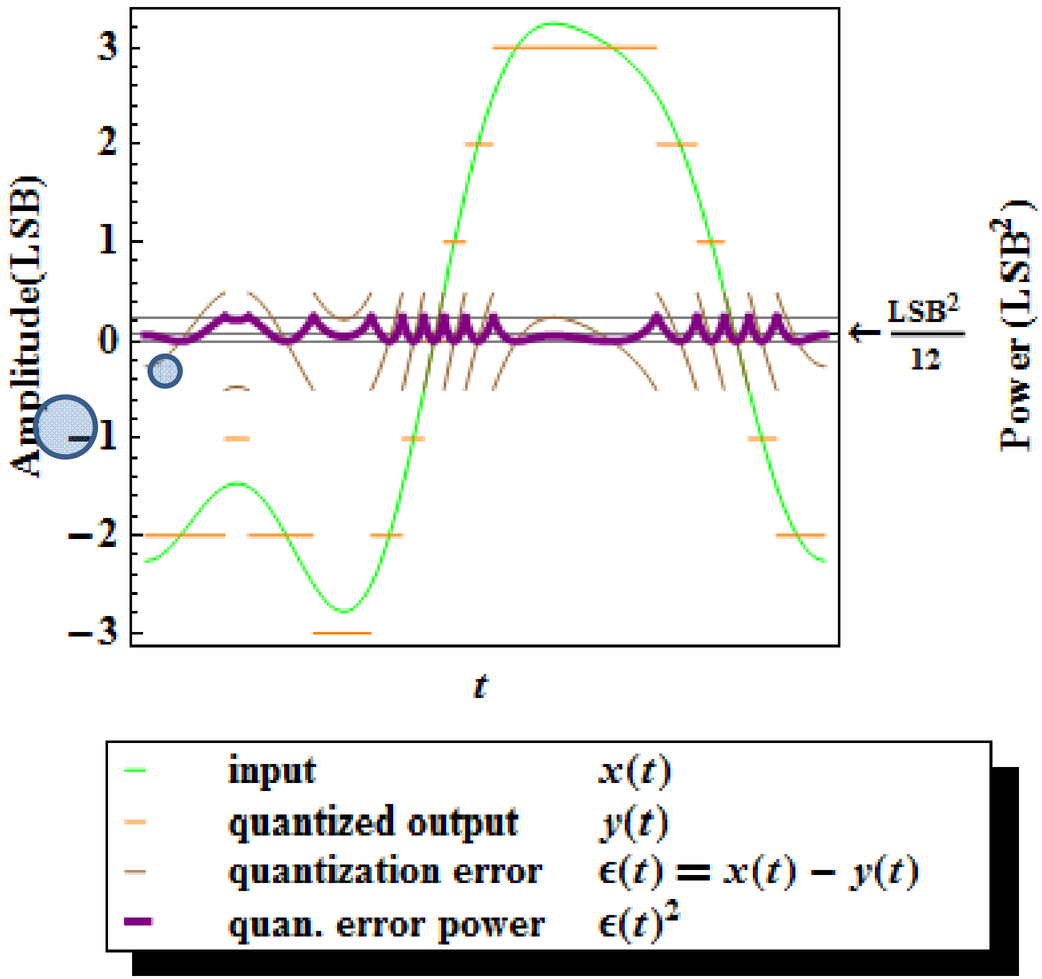




Error cause quantization noise

- Noise power = variance of error ε





Quantization noise power = 1/12 LSB^2

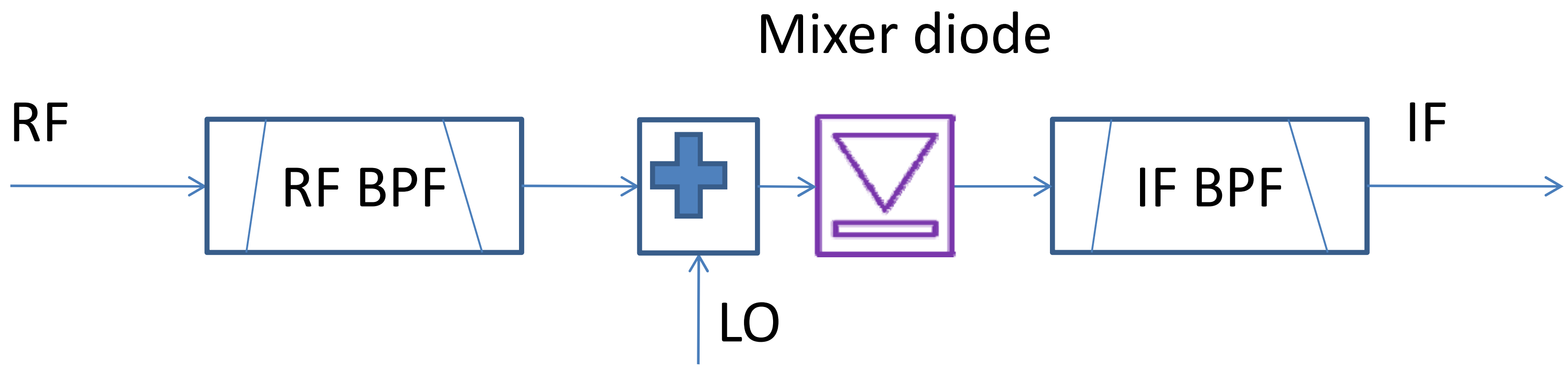
- If
  - The quantization step width is small enough compared to the input amplitude
  - And the quantizer has large enough number of steps to cover the input amplitude

Anyway,
Quantization noise power ≠ 0

- Can it be possible to make **Analog-to-Digital Conversion** **noiseless**?

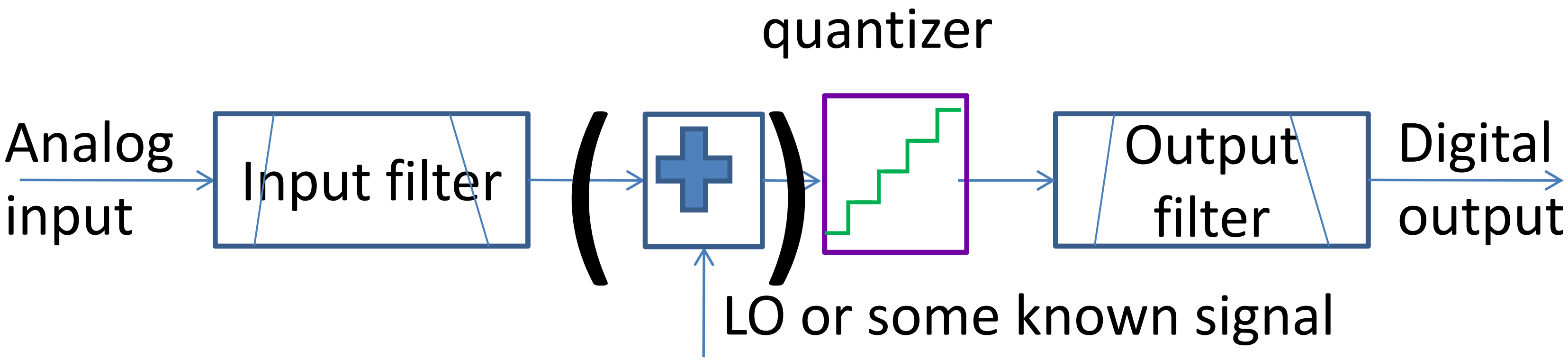
- To answer that, consider other examples of non-linear circuits.

Typical Frequency Converter with a Mixer Diode



- A mixer diode is surely non-linear.
- It introduces harmonics and inter-modulations.
- But they are blocked by the IF Band-Pass Filter!
- And we get distortion-less/noise-less IF.

Noise Reduction in Analog-to Digital Converter



- A quantizer is surely non-linear.
- It introduces harmonics and inter-modulations.
- But they will be blocked by the Output Filter!
- And we get distortion-less/noise-less Digital output with increased word-length.

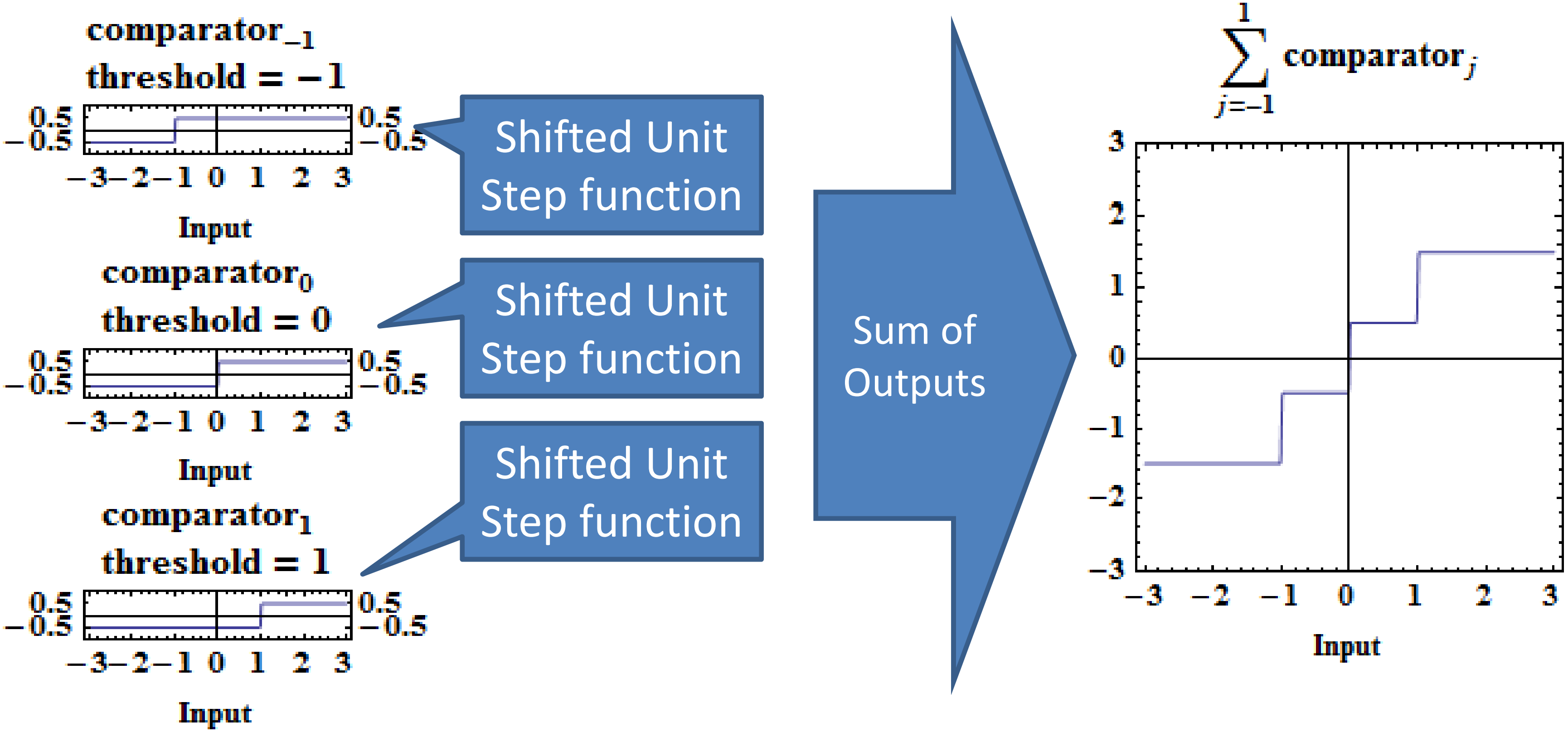
Is there any predecessor eliminating distortion by a non-linear effect?

- Yes, there is. It was patented 70 years ago.
- "AC biasing in the magnetic audio recorder"
- By adding CW ('LO') of several tens of kHz, the patent reduced the distortion which was caused by non-linearity of magnetic material.
- SONY utilized it to energize its voyage to a distinguished company.

To Reduce Noise in
Analog-to Digital Converter

- Think in the frequency domain.
- To do so…
- Analytical treatment to distinguish harmonics and inter-modulations is needed.

Break down an N-level quantizer into a sum of N-1 2-level quantizers.



- Then we only need to analyze a 2-level quantizer.

i.e.,

- Output of an n-level quantizer is as follows.

$$\textbf{output} = \sum_{j=1}^{n} \Delta_j \left(\theta\left(\frac{\textbf{input}}{2\sqrt{\langle|\textbf{input}|^2\rangle}} - \Theta_j \right) - \frac{1}{2} \right), \textbf{ where } \theta(x) \textbf{ is the Unit Step function}$$

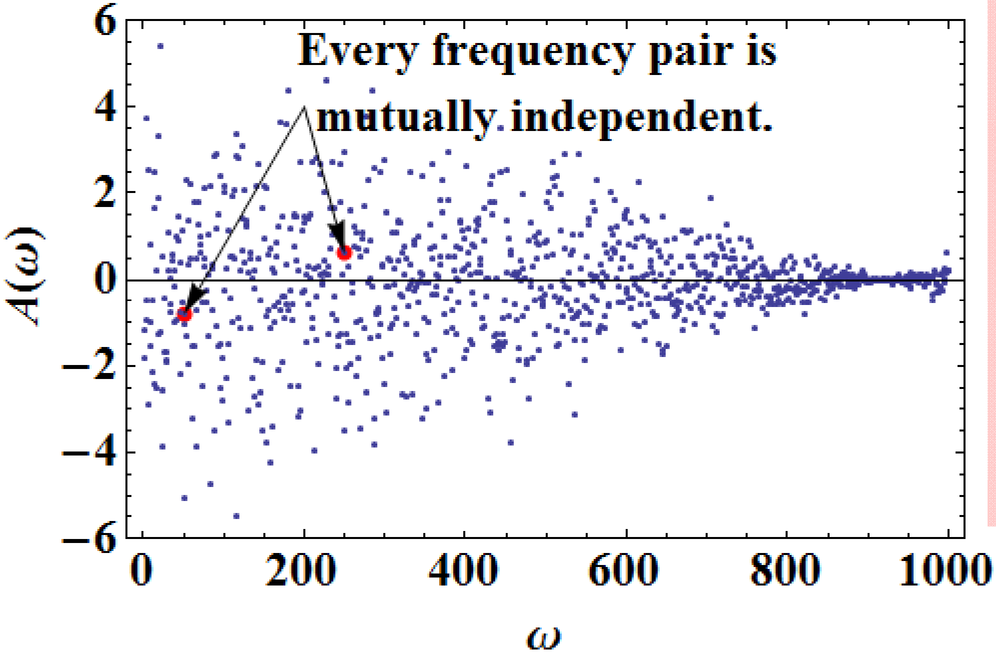
$$\Theta = \frac{\textbf{threshold}}{2\sqrt{\langle|\textbf{input}|^2\rangle}}$$

Assume that the signals are 'pink noise'.

Definition: Call $a(t)$ a 'pink noise', if $A(\omega)$ meets following condition when $A(\omega)$ is the Fourier transform of $a(t)$ and $\delta(i) = \begin{cases} 1 & i = 0 \\ 0 & \text{otherwise} \end{cases}$:

$$\forall_{\{\omega,\Omega,i,j\},i\neq 0 \vee j\neq 0} \left\langle A(\omega)^i \left(A(\Omega)^*\right)^j \right\rangle = \delta(i-j)\,\delta(\omega-\Omega) \left\langle \left(|A(\omega)|^2\right)^i \right\rangle$$

Schematics Drawing for 'Pink Noise' Spectrum



This 'pink noise' assumption is very natural to macroscopic objects where many independent oscillators are in the sight.

Consider a time series *a(t)* of a 'pink noise' *A(ω),* *a(t)* obeys the Normal Distribution.

- ∵） *a(t)* is Fourier Transform of *A(ω)*, i.e., at each *t* it is a sum of *phase***A(ω)*.
- Remember that PDF(a+b) is convolution of PDF(a) and PDF(b).
- The Central Limit Theorem confirms the conclusion.

- After lengthy calculations:
  - Breaking down an N-level quantizer into a sum of N-1 2-level quantizers,
  - Considering a Unit Step function as a limit of Error function when steepness $\rightarrow\infty$,
  - Using Taylor expansion of Error function,
  - Assuming ‘pink noise’ inputs,
  - …

The correlation function of the output or the quantization correction is:

$$r(\rho, \phi, \Theta_i, \Theta_j, \Delta_i, \Delta_j, \omega, \tau) = \sum_{K=-\infty}^{\infty} \exp(i\, K\, (\omega\, \tau + \phi)) \sum_{\kappa=\left|\frac{K}{2}\right|}^{\infty} \sum_{i=1}^{n_A} \sum_{j=1}^{n_B} \frac{|\rho(\tau)|^{2\,\kappa}\, \Delta_i\, \mathrm{coef}(\kappa, \Theta_i)\, \Delta_j\, \mathrm{coef}(\kappa, \Theta_j)}{\pi\, \left(-\frac{K}{2} + \kappa\right)!\, \left(\frac{K}{2} + \kappa\right)!}$$

r : output correlation function between ADC−A and ADC−B

ρ : input correlation coefficient between ADC−A and ADC−B

ϕ : input correlation phase between ADC−A and ADC−B

n_A, n_B : number of thresholds for ADC−A and ADC−B

Δ_i : output step hight for i−th level

Θ_i : normalized input threshold for i−th level

$$\text{output} = \sum_{j=1}^{n} \Delta_j \left(\theta\left(\frac{\text{input}}{2\sqrt{\langle|\text{input}|^2\rangle}} - \Theta_j\right) - \frac{1}{2}\right), \text{ where } \theta(x) \text{ is the Unit Step function}$$

κ : order of inter−modulations

K : order of harmonics

ω : 'center' frequency of the input

τ : lag argument of the correlation

where

$$\mathbf{coef}(\kappa, \Theta) = \begin{cases} -\dfrac{\Theta\,(-1)^{\kappa}\,\Gamma\left(\kappa+\frac{1}{2}\right)\,{}_1F_1\left(\kappa+\frac{1}{2};\frac{3}{2};-\Theta^2\right)}{\sqrt{\pi}} & \kappa \bmod 1 = 0 \\ \dfrac{(-1)^{\kappa-\frac{1}{2}}\,\Gamma(\kappa)\,{}_1F_1\left(\kappa;\frac{1}{2};-\Theta^2\right)}{2\sqrt{\pi}} & \kappa \bmod 1 = \frac{1}{2} \end{cases}$$

${}_1F_1(a; b; z)$ **: Kummer confluent hypergeometric function of the first kind**

κ **: order of inter−modulations**

Θ **: normalized input threshold :** $\Theta = \dfrac{\text{threshold}}{2\sqrt{\langle|\text{input}|^2\rangle}}$

Important thing is:

$$r\left(\rho, \phi, \Theta_i, \Theta_j, \Delta_i, \Delta_j, \omega, \tau\right) =$$

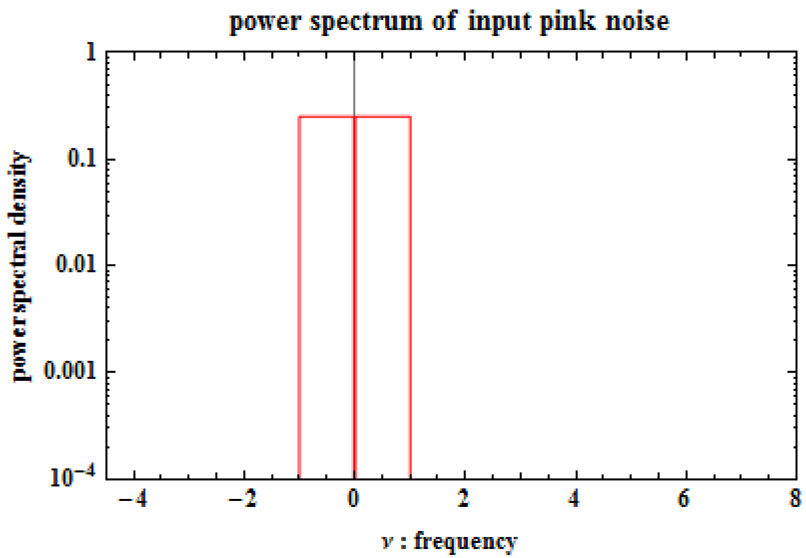
$$\sum_{K=-\infty}^{\infty} \exp(i\, K\, (\omega\, \tau + \phi)) \sum_{\kappa=\left|\frac{K}{2}\right|}^{\infty} \sum_{i=1}^{n_A} \sum_{j=1}^{n_B} \frac{|\rho(\tau)|^{2\kappa}\, \Delta_i\, \mathrm{coef}(\kappa, \Theta_i)\, \Delta_j\, \mathrm{coef}\left(\kappa, \Theta_j\right)}{\pi \left(-\frac{K}{2} + \kappa\right)!\, \left(\frac{K}{2} + \kappa\right)!}$$

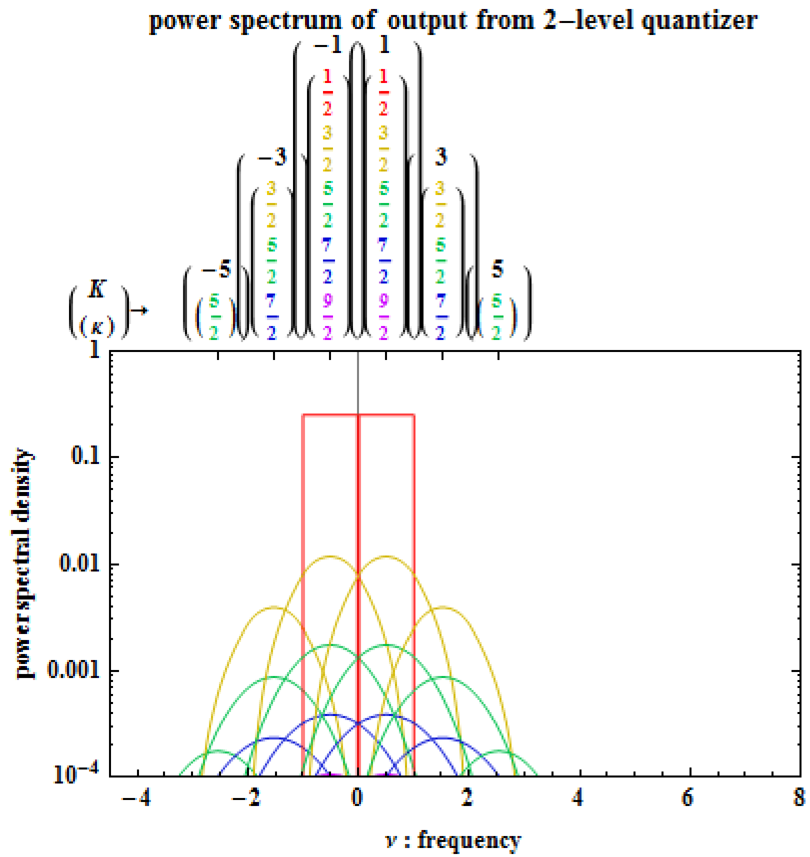
r : output correlation function between ADC−A and ADC−B

- We have succeeded to analytically derive the output correlation function for any ADC as a sum over:
  - K: harmonics order
  - κ : inter-modulation order

Example of 2-level quantizer spectrum

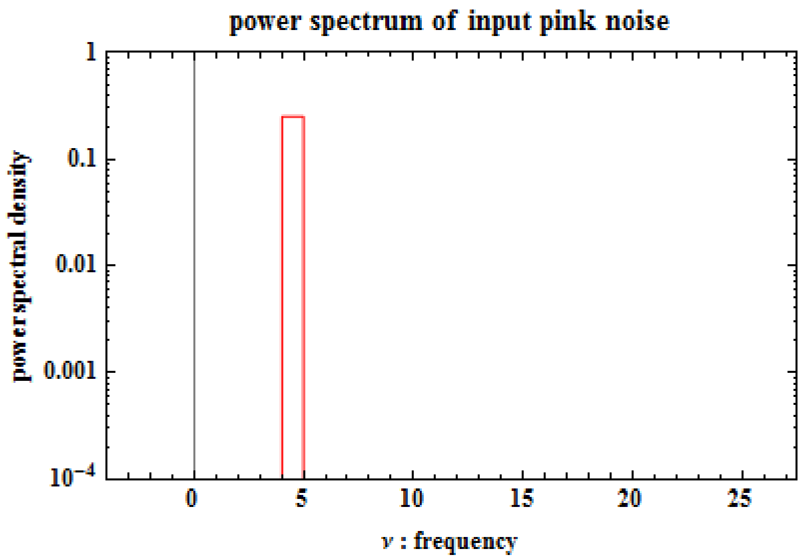
- Baseband input causes many harmonics and inter-modulations in the signal band.

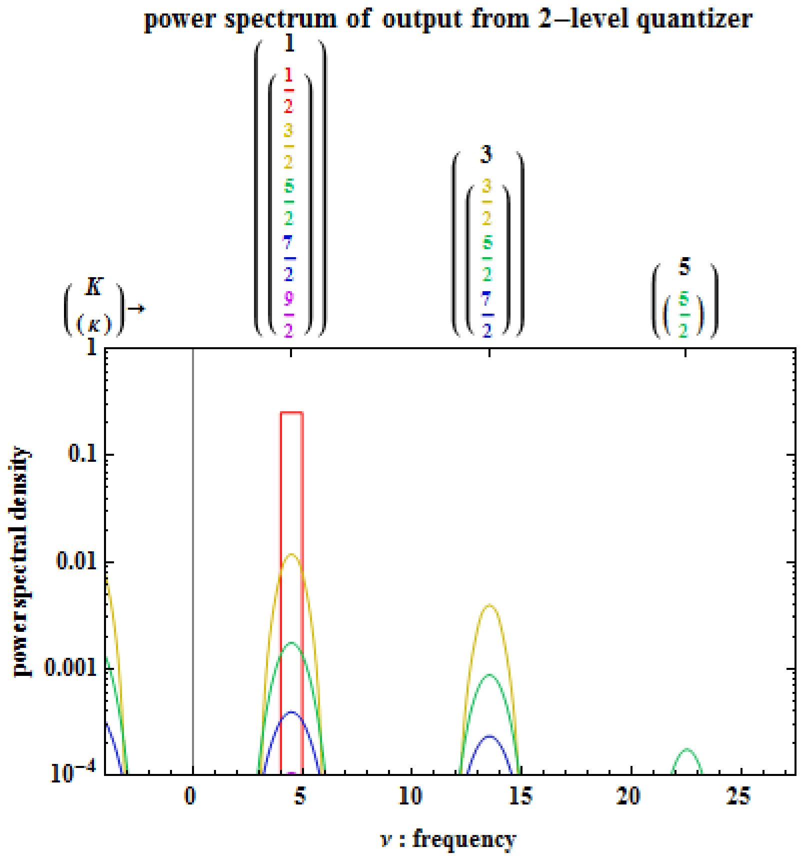




Example of 2-level quantizer spectrum

- Non-baseband input reduces harmonics and inter-modulations in the signal band.





Frequency domain approach has succeeded in deeper understandings

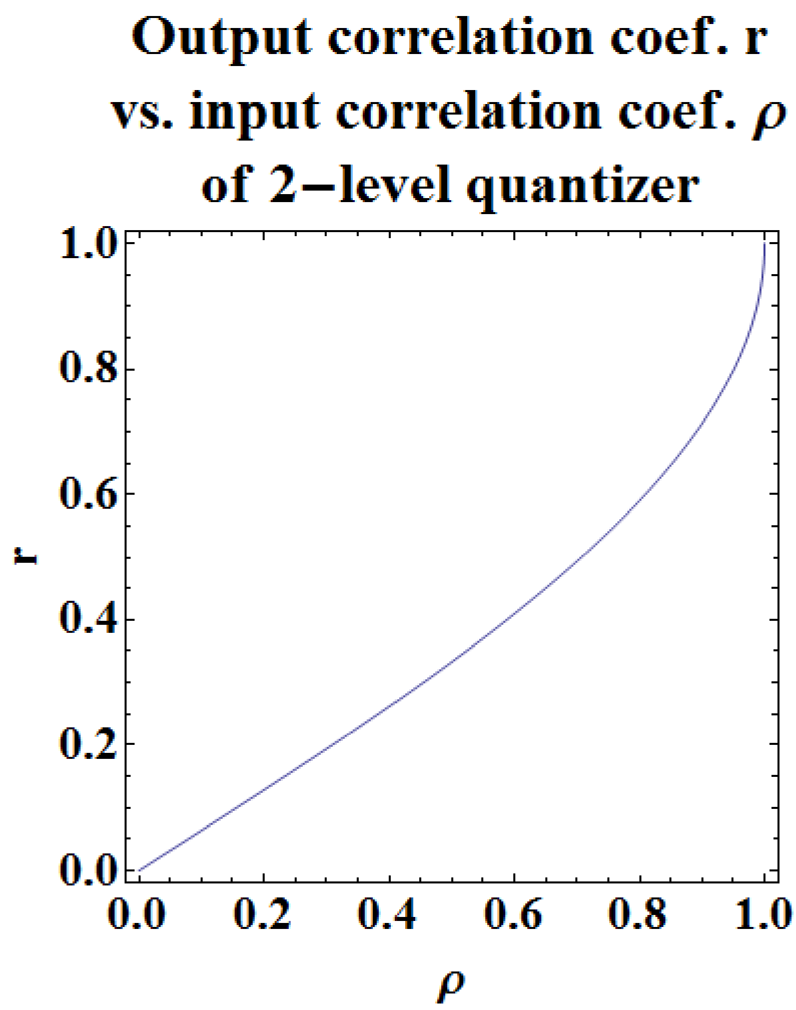
- The correction is conventionally understood via time domain as integral of joint probabilities between the digitized signals.
- We have analytically showed this correction in general form via frequency domain approach.

An Introduction:
Small-bit ADC in radio astronomy

- In radio astronomy,
- to have receiving bandwidth as wide as possible,
- ADCs with very small number of quantization levels (usually 2-8 levels, i.e., 1-3 bits) have been used.
- Therefore it is very important to develop means to minimize noises and/or spectrum deformation caused by an ADC.

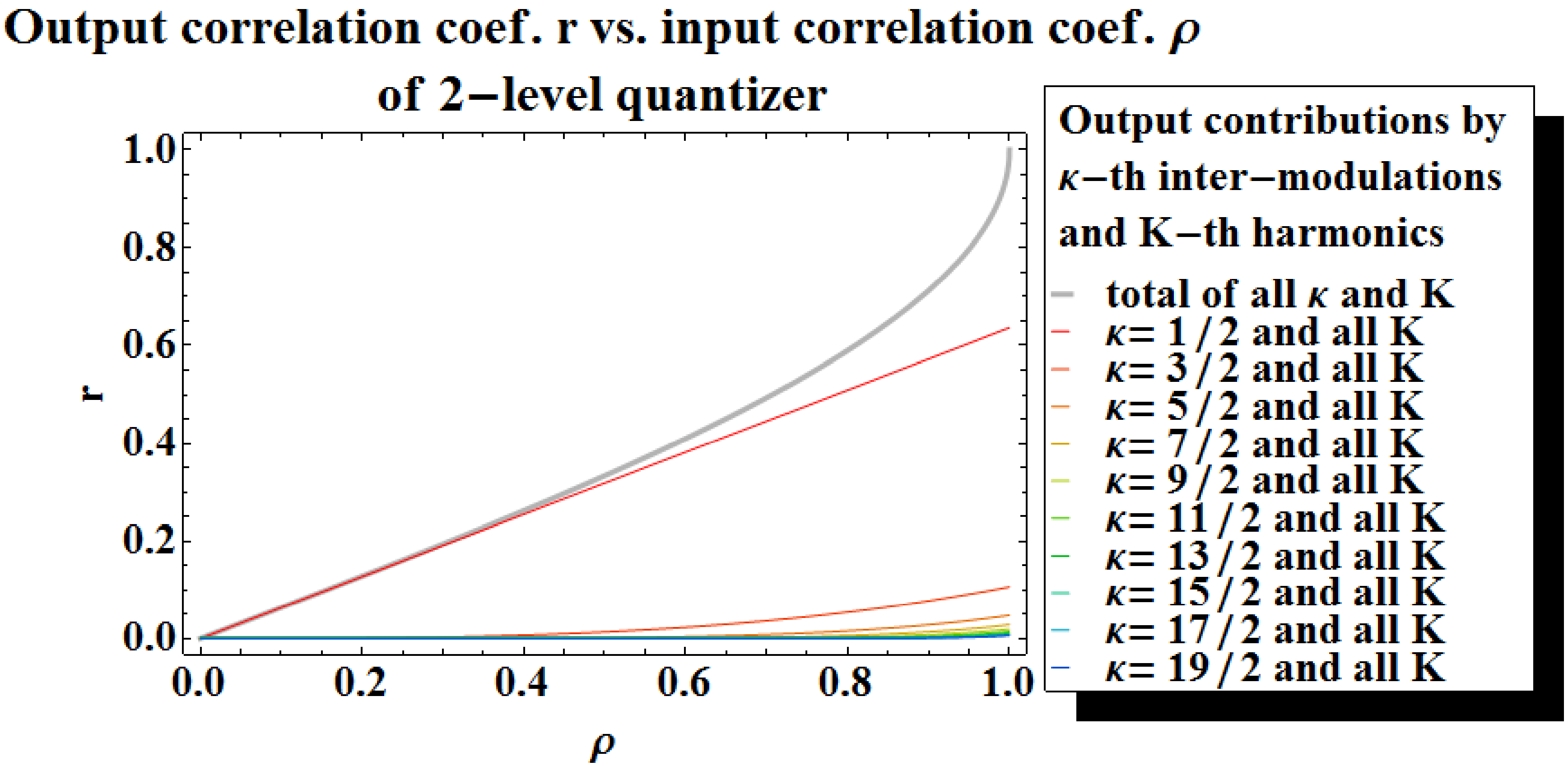
The Van Vleck correction or the quantization correction

- Correct
  - the observed digitized correlation coefficient
  - to the analog one.
- For an example, in a two level quantizer case, the former is proportional to arcsine of the latter.
  - A Nyquist-sampled baseband case



Sudden rise in the correction curve when the correlation coefficient approaches to unity is:

- Caused by the higher κ and K order terms.



Sudden rise in the correction curve when the correlation coefficient approaches to unity is:

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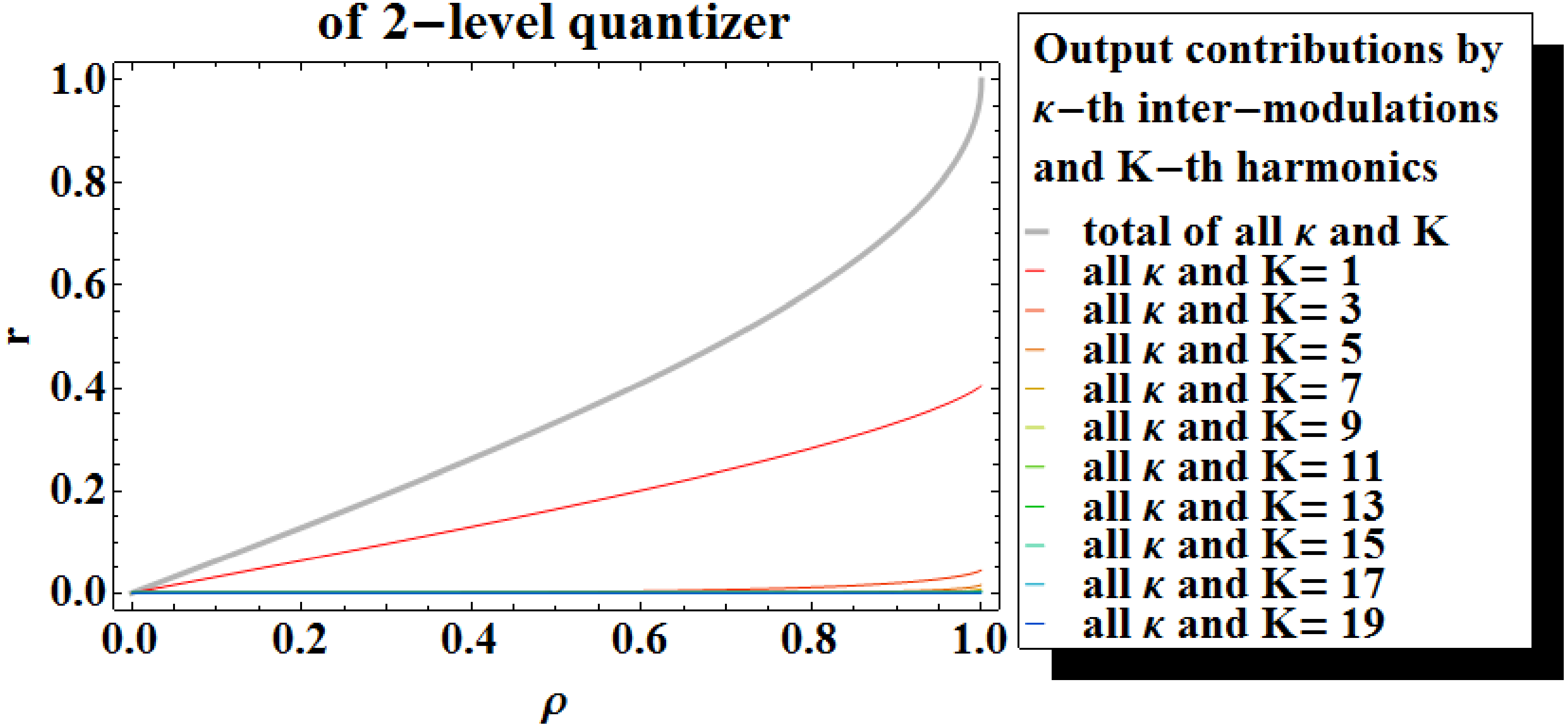
Output correlation coef. r vs. input correlation coef. ρ of 2–level quantizer



Summary

- Via frequency domain:
- A way to realize ‘noiseless/reduced-noise’ ADC was demonstrated.
- Analytical expression of ADC output correlation function was given.
- The expression was given as a sum over harmonics order and inter-modulation order.
- The expression is very convenient for filtering.